COSC 341
Theory of Computing
Lecture 21
More reductions from 3-SAT to graph problems
(3-COLOURING and HAMILTON-CYCLE)

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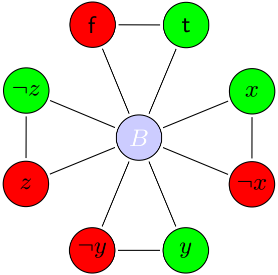
3-SAT to 3-COLOURING

Step 1: Define a "core" graph, in which each 3-colouring corresponds to a unique truth assignment for the 3-SAT instance. If there are k variables, we create $k+1$ triangles in the graph, sharing a common "base" vertex. One triangle has its other two vertices labelled t and f.

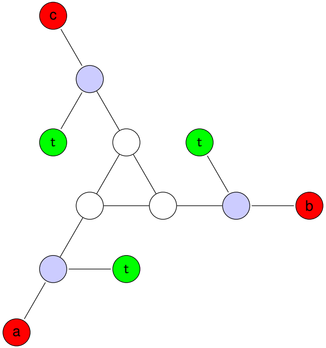
Example for the 3-variable case (the 3-colouring corresponds to $x = \mathtt{t}$, $y = \mathtt{t}$ and $z = \mathtt{f}$):



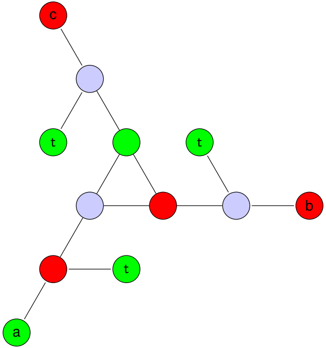
3-SAT to 3-COLOURING (continued)

Step 2: Add "gadgets". For each clause $a \vee b \vee c$ in the 3-SAT instance (where a, b and c are literals), interconnect the vertices for these literals and t in the core graph as shown below:

All literals false. Impossible to 3-colour



Some literals true. A 3-colouring exists.



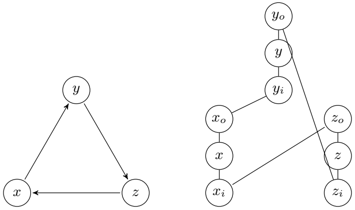
3-SAT to HAMILTON-CYCLE

It is convenient to do this in two steps:

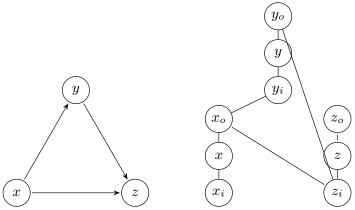
- First reduce 3-SAT to DIRECTED-HAMILTON-CYCLE (determining whether a directed graph has a Hamilton cycle).
- Then reduce DIRECTED-HAMILTON-CYCLE to HAMILTON-CYCLE.

We will show the second step first. Given a directed graph G_d, we construct an undirected graph G_u. Each vertex v in G_d is replaced by three vertices in G_u: v_i, v and v_o (the "i" and "o" mean "in" and " out). We add edges (v_i, v) and (v, v_o). An edge (v, w) in G_d becomes (v_o, w_i) in G_u.

There is a Hamilton cycle in one precisely when there is a Hamilton cycle in the other (see Notes 18).



A Hamilton cycle exists



No Hamilton cycle exists

3-SAT to DIRECTED-HAMILTON-CYCLE

Example: the directed graph produced for a formula including the clause $x \vee \neg y \vee z$. Nodes and edges related to other clauses are not shown. See Notes 18 for details.