COSC341 TUTORIAL 6, SOLUTIONS

Doing some exercises about equivalence relations in general and then the particular versions associated to languages and DFAs. As usual, $\Sigma = \{a, b\}$ unless otherwise specified.

1. *Which of the following are equivalence relations on the set* $\mathbb{P} = \{1, 2, 3, \dots\}$ *of positive integers? For those which are, describe the equivalence classes, and determine how many there are.*

   - $a \sim b$ *means "a is a divisor of b".*
     No, not symmetric, e.g., $2 \sim 4$ but $4 \not\sim 2$.
   - $a \sim b$ *means "the prime divisors of a and b are the same"*
     Yes, typical "the same" type relation. The equivalence classes correspond to finite subsets of the primes and consist of all those numbers that can be written as products of elements of the subset using each element at least once (but possibly many times).
   - $a \sim b$ *means "some digits (base 10) of a can be deleted to give b"*
     No, obviously not symmetric.
   - $a \sim b$ *means "the sum of the digits (base 10) of a and b are the same"*
     Yes, there is an equivalence class for each non-negative integer, n, whose elements are those integers whose digit sums equal n.
   - $a \sim b$ *means "either both a and b are less than 100, or both are greater than or equal to 100"*
     Yes, two classes - those less than 100 and those greater than or equal to 100.
   - $a \sim b$ *means "$a - b$ is a multiple of 3 or of 5"*
     No, not transitive $0 \sim 3 \sim 8$ but $0 \not\sim 8$.
   - $a \sim b$ *means "$a - b$ is a multiple of 3 and of 5"*
     Yes, because this means that the difference is a multiple of 15 (more generally the "conjunction" of two equivalence relations is an equivalence relation). The equivalence classes correspond to remainders modulo 15.

2. *Suppose that a DFA,* $\mathbf{A}$ *has two different garbage states. Are words that lead to one state-equivalent to words that lead to the other? Are they suffix-equivalent?*

   They are not state-equivalent since they lead to different states. They are suffix-equivalent since the corresponding suffix languages are both empty.

3. *We will generate some small DFAs randomly and figure out the state- and suffix- equivalence relations.*

   Just do it!

4. *Use the Myhill-Nerode theorem to show that none of the following languages are regular:*

   - $L = \{a^n b^n \, : \, n \geqslant 0\}$

     Consider the suffix language of a^k for any $k \geqslant 0$. Among its elements is b^k – in fact this is the only element of b^* that belongs to its suffix language. This implies that for $k \neq j$ the suffix languages of a^k and a^j are different and, in particular, there are infinitely many different suffix-languages for L, so by Myhill-Nerode, L is not regular.

   - *L is the set of palindromes – words that equal their own reverses, e.g., aba, $babab$, a, etc.*

     A small modification of the above works. For any $k \geqslant 0$ the only word from a^* that belongs to the suffix language of $a^k b$ is a^k. In fact, much more is true - no two words have the same suffix language.

5. *Use the Myhill-Nerode theorem to show that the following language is regular:*

$$L = \{a^n b^k \, : \, 17 \text{ is a divisor of } k - n\}$$

   We need to show that the suffix-equivalence relation for L has only finitely many classes. First observe that if a word w is not in $a^* b^*$ then no suffix can be appended to give a word in L. So, all such words belong to a single equivalence class. Now, for a word in $a^* b^*$, write it as $a^s b^t$. Suppose first that $t > 0$. Then the only allowed suffixes are words b^m for which $m + t - s$ is a multiple of 17. But, the set of such m depends only on the remainder when we divide $t - s$ by 17. So, there are 17 such classes. Finally consider words a^s. The allowed suffixes are $a^p b^m$ such that $m - p - s$ is a multiple of 17. But, again, this set only depends on the remainder when we divide s by 17 so that gives 17 more classes. It seems that the total number of equivalence classes is 35 and so L is regular.