1 Tutorial problems

0. *Draw some graphs. Make sure you're happy with notions like walk, path, cycle etc.*

   Assumed done!

1. *What is the least number of edges that a connected graph with n vertices can have?*

   Consider an equivalence relation on the vertices of a graph given by $v \sim w$ if there is a walk from v to w in the graph (check that this is an equivalence relation - this amounts to noting that "going nowhere" is a walk, that "walking in the opposite direction" is possible, and that "joining walks together" is possible)[1]

   Now consider adding an edge $\{v, w\}$ to a graph – how does this change the relation $\sim$? If the edge connects two vertices that were already equivalent, then not at all (we've just introduced a shortcut between two places when we already knew how to walk from one to the other). If they weren't then the two classes that contained v and w respectively have now merged into a single class, i.e., the number of $\sim$-equivalence classes is reduced by one.

   If we start with no edges at all then there are no non-trivial $\sim$-equivalences, i.e., n equivalence classes. To get this down to 1 therefore requires at least $n - 1$ edges (and this is sufficient - any path or indeed any tree on n vertices has $n - 1$ edges and is connected).

2. *(*) What is the greatest number of edges that a graph on n vertices containing no triangle (clique of size 3) can have?*

   The answer is $\lfloor n/2 \rfloor \lceil n/2 \rceil$ (where $\lfloor x \rfloor$ is the greatest integer less than or equal to x, and $\lceil x \rceil$ is the least integer greater than or equal to x). This also happens to equal $\lfloor n^2/4 \rfloor$

   This is *Mantel's theorem* (1907) an instance of a more general result called Turán's theorem.

   Here's one way to prove it. We're going to show that, among the triangle-free graphs on n vertices having the maximum possible number of edges there's one of a specific type. Namely, consider any triangle free graph G. Choose a vertex x in G which has at least as many neighbours as any other vertex. Now, if y is any non-neighbour of x we could remove all the edges from y to its neighbours, and replace them with edges from y to x's neighbours. Since x's neighbours contain no edges (we had not triangles) and since xy was not an edge, this creates no triangles. Moreover, by our choice of x the number of edges does not go down. Doing this for all the non-neighbours of x we get a *complete bipartite graph*. We have one independent set of vertices (including x) and another (the neighbours of x) and all possible edges between them. So we can conclude that, among all the the triangle-free graphs on n vertices having the maximum possible number of edges there's one that is complete bipartite.

   If there are k vertices in the part containing x then the total number of edges is $k(n-k)$. It's easy to check that this is maximised when $k = \lfloor n/2 \rfloor$ and then a bit of arithmetic gives the formula above (the odd case is slightly messy).

[1] In 2020 thinking of vertices as people, edges as "contacts" and $\sim$-equivalence classes as "bubbles" makes a lot of sense!

3. *Show that in a graph G if there is a walk from v to w then there is also a path from v to w.*

   If a walk exists, then there's a shortest walk. A shortest walk can't contain a duplicated vertex since we could just skip the part between the first and last occurrence of that vertex and shorten it. So a shortest walk between two vertices is a path.

4. *Find a graph with five vertices that has no clique nor independent set of size 3. Doe such a graph exist with six vertices?*

   A cycle of 5 vertices has no clique nor independent set of size 3. Suppose we have a graph of six vertices. Choose any vertex v. Since there are 5 other vertices, either v is adjacent to at least three of them, or independent from at least three of them.

   In the first case, say v is adjacent to a, b, c. Then if there is any edge among those three, we get a triangle, while if there is none, we get a three element independent set.

   Similarly, in the second case, say v is independent of a, b, c. This time if there's any independent pair among those three we get a three element independent set, while if all three are connected by edges we have a triangle.

   This is the first interesting case of Ramsey's theorem.

5. *Suppose that we have a polynomial-time reduction from* PROB1 *to* PROB2.

   (a) *If the time required to convert an instance of size n is bounded by n^4, then how large an instance of* PROB2 *might we get from an instance of size n for* PROB1*?*

   Since it takes at least as much time to write down an instance as its size, there's a bound of n^4 on the size of the instance we get.

   (b) *If* PROB2 $\in \mathbf{P}$ *and the time required to resolve an instance of size n of* PROB2 *is bounded by n^3, what is an upper bound for the time required to resolve an instance of* PROB1 *through the reduction?*

   We might need to solve an instance of size n^4 which could take $(n^4)^3$, i.e., we get a bound of n^{12} – still polynomial though.

6. *Confirm that* HAS-PATH, IS-CONNECTED, HAS-CYCLE, *and* 2-COLOURING *are in* $\mathbf{P}$.

   All are effectively resolved by breadth-first search (BFS) in a graph. Since a BFS examines each edge of a graph only once (or possibly twice - once from each end) the time-complexity is polynomial. In fact all four are essentially resolved by constructing the minimum distance spanning tree from any initial vertex v (as in Dijkstra's algorithm).

   This resolves the first two directly ("does the other vertex occur", "do all vertices occur") the third one implicitly ("is there any vertex with an edge not in the tree") and the fourth relatively directly ("are there any edges between vertices at odd distance from v and at even distance from v?). In the latter two cases we may need to carry out the algorithm on each of the connected components of G separately (if G is not connected.)