1 Tutorial problems

1. *Which of the following formulas are satisfiable?*

   (a) $(x \vee y) \wedge (\neg x \vee \neg y)$
   $x = \mathsf{t}, y = \mathsf{f}$ or vice versa satisfies this.

   (b) $(x \vee y) \wedge (\neg x \vee \neg y) \wedge (\neg x \vee y)$
   This time only $x = \mathsf{f}, y = \mathsf{t}$ works.

   (c) $(x \vee y) \wedge (\neg x \vee \neg y) \wedge (\neg x \vee y) \wedge (x \vee \neg y)$
   Nothing works, i.e., unsatisfiable.

   (d) $(\neg a \vee b \vee c) \wedge (a \vee \neg b) \wedge (a \vee \neg c)$
   Lots of things work, e.g., $a = \mathsf{t}$ satisfies the last two clauses so we can then take $b = \mathsf{t}$ or $c = \mathsf{t}$. If $a = \mathsf{f}$ we need to satisfy the second and third for which we can use $b = c = \mathsf{f}$.

2. *Verify that the following pairs of formulas are equivalent:*

   (a) $a \wedge (b \vee c)$ and $(a \wedge b) \vee (a \wedge c)$,
   This could be done with an eight line truth table, but can also be done in English. The first of the two formulas is true if and only if a is true, and at least one of b, c is true. But that's the same as saying at least one of "both a and b" or "both b and c" is true, which is the second formula.

   (b) $a \vee (b \wedge c)$ and $(a \vee b) \wedge (a \vee c)$,
   Similar to the preceding one. For the first formula to be true at least one of a, or "both b and c" must be true. If a is true, then so is the second formula, and if both b and c are true then so is the second formula. For the first formula to be false, a must be false and one of b or c must be false. But then one of the conjuncts in the second formula is false, hence it is too.

   (c) $\neg(a \vee b)$ and $\neg a \wedge \neg b$,
   Neither a nor b is true is the same as a is not true and b is not true.

   (d) $\neg(a \wedge b)$ and $\neg a \vee \neg b$.
   Not both a and b are true is the same as saying at least one fo them is false.

   But remember, in a pinch, you can always just write out the truth tables and check that they match.

3. *Show that for any Boolean formula $\Phi(x_1, x_2, \ldots, x_k)$ there is an equivalent formula $\Psi(x_1, x_2, \ldots, x_k)$ in conjunctive normal form.*

   I think I'm going to do this in a slightly non-standard way (the standard way would be to do a recursion but one of the cases gets a bit messy).

   Consider any map from $F : \{\mathsf{t}, \mathsf{f}\}^k \to \{\mathsf{t}, \mathsf{f}\}$, i.e., any function that takes k-tuples of truth values (a truth assignment) and produces truth values. Suppose that $F(v_1, v_2, \ldots, v_k) = \mathsf{f}$. Consider the clause:

$$y_1 \vee y_2 \vee \cdots y_k$$

where $y_i = x_i$ if $v_i = \mathsf{f}$ and $y_i = \neg x_i$ if $v_i = \mathsf{t}$. Then this clause is f exactly (and only) for the truth assignment $x_i \mapsto v_i$. Therefore, the conjunction of these clauses over all tuples on which F is false is satisfied exactly by the truth assignments that make F true. In other words that CNF formula is "equivalent" to F.

We've actually shown a bit more which is that any map F such as the above represents the truth values of some CNF formula.

4. *Find a CNF-formula equivalent to*

$$(a \wedge b) \vee (c \wedge d) \vee (e \wedge f)$$

This should suggest why we can have an exponential blow-up in size when we convert to CNF (though it doesn't prove it as it stands – that's a little technical, since you need to provide a lower bound on the length of the shortest CNF-formula equivalent to a given formula.

$$\bigwedge \left\{ \begin{array}{c} a \vee c \vee e \\ a \vee c \vee f \\ a \vee d \vee e \\ a \vee d \vee f \\ b \vee c \vee e \\ b \vee c \vee f \\ b \vee d \vee e \\ b \vee d \vee f \end{array} \right\}$$

Notice that we wound up with $8 = 2^3$ clauses from three conjuncts of size 2 – this exponential pattern can be further extended.

As I said, the proof that this exponential blow up is needed is a bit more technical.