1 Tutorial problems

1. *How do you search for a satisfying assignment of a 3-*SAT *instance if you have a* $(\leqslant 3)$-SAT *black box?*

   Set a variable to t. See if the remaining clauses (those containing its negation or missing it entirely) are still satisfiable. If so, continue. If not, set it to false and continue.

2. *How do you search for a satisfying assignment of a 2-*SAT *instance if you have a 3-*SAT *black box? The difference is the requirement that 3-*SAT *clauses contain exactly three literals rather than 2.*

   You have to do the 'add dummy variables' trick to the size 2 clauses (i.e. replace $a \vee b$ say by $a \vee b \vee c$ and $a \vee b \vee \neg c$ where c is a fresh variable). That trick was discussed in the introduction to k-SAT in Lecture 20.

3. *How can you find a 3-colouring of a graph given a black box for the 3-colouring decision problem?*

   If the graph you're interested in is not 3-colourable, fail. Otherwise, add a triangle of new vertices. For each original vertex of the graph, connect two vertices of the triangle to it. Ask 'are we still three-colourable'. If so, continue. If not, choose another pair until we find a pair we can connect. After doing this over all the original vertices we have the original graph with every vertex connected to two of the vertices of the triangle, and can colour it by choosing the same colour for all vertices connected to the same pair.

4. *Consider a clause of three distinct literals. If we choose a random truth assignment for each variable independently (and with a* $1/2$ *chance of either* t *or* f*), then what is the probability,* p*, that we will satisfy that particular clause?*

   $\frac{7}{8}$ since there is only one assignment that fails to satisfy the clause.

5. *(*) Given the previous problem, if we have* k *clauses there must be a truth assignment that satisfies at least* kp *of them (because this is the average number satisfied by any given assignment). Show that there is a polynomial-time deterministic algorithm that determines such an assignment.*

   Suppose we set a variable to t. This satisfies a certain number S of clauses, leaves a certain number N of size 2 clauses (those with its negation) and a certain number, R, of remaining clauses. The expected number of satisfied clauses when we randomly set the remainder of the variables is:

   $$S + \frac{3}{4}N + \frac{7}{8}R$$

   If this is more than $\frac{7}{8}$ of the total clauses, consider this part of the assignment fixed and move to the next variable. If it is less, then by additivity of expectation, the expected number of satisfied clauses when we make the variable false is more than $\frac{7}{8}$ of the total number of clauses. So, make it false and move to the next variable.

   (For each variable we can determine in polynomial time whether, for a given assignment of the preceding variables we are better off in expectation by making it t or f. So, when we assign them all in this way we must get at least the original expectation).